Strictly Semistable Quasimaps on the Wall

Sara Stephens

Cornell University

ϵ-Stable Quasimaps to $\mathbb{P}^N$

Definition. A n-pointed prestable **quasimap** to $[\mathbb{C}^{N+1}/\mathbb{C}^*]$ of genus g and degree d consists of data $(C, x_1, ..., x_n, L, \{s_i\}_{i=0}^N)$ where

- C is a connected nodal curve of genus g and marked points $x_1, ..., x_n$
- L is a degree d line bundle over C with $s_0, \dots s_N \in \Gamma(C, L)$
- the set of zeroes of $(s_0, \dots s_N)$ is finite and disjoint from the nodes and the markings on C

Such a quasimap is **ϵ-stable** for $\epsilon \in \mathbb{Q}_{>0}$ if

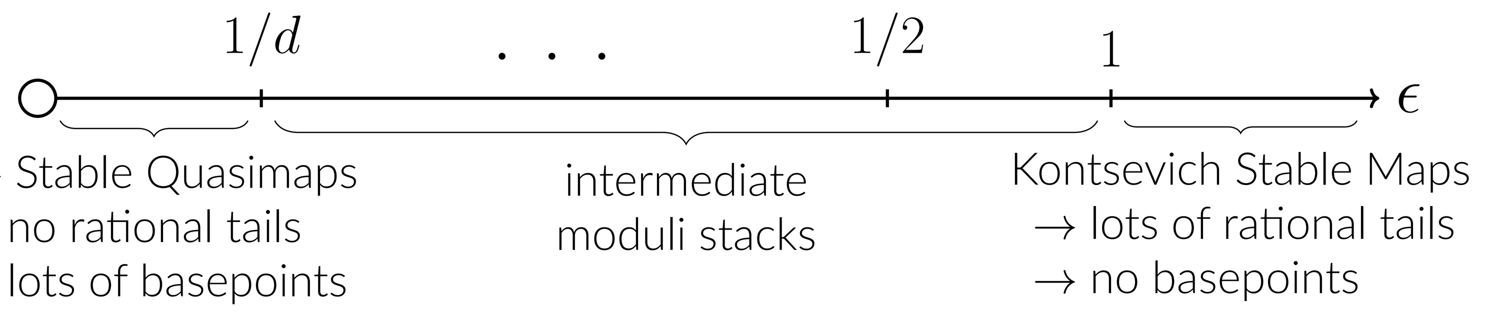
1. $\omega_C(x_1 + \dots + x_n) \otimes L^\epsilon$ is ample and
2. For every point $x \in C$ we have $\ell(x) \leq 1/\epsilon$, where $\ell(x)$ is the *length* of the point $x \in C$, given by

$$\ell(x) := \mathrm{length}_x(\mathrm{coker}(\mathcal{O}_C^{\oplus N+1} \to L)).$$

Theorem [CFKM14] . The stack $\mathrm{QMap}^\epsilon_{g,n}(\mathbb{P}^N, d)$ is a proper Deligne-Mumford stack of finite type.

Wall and Chamber Structure

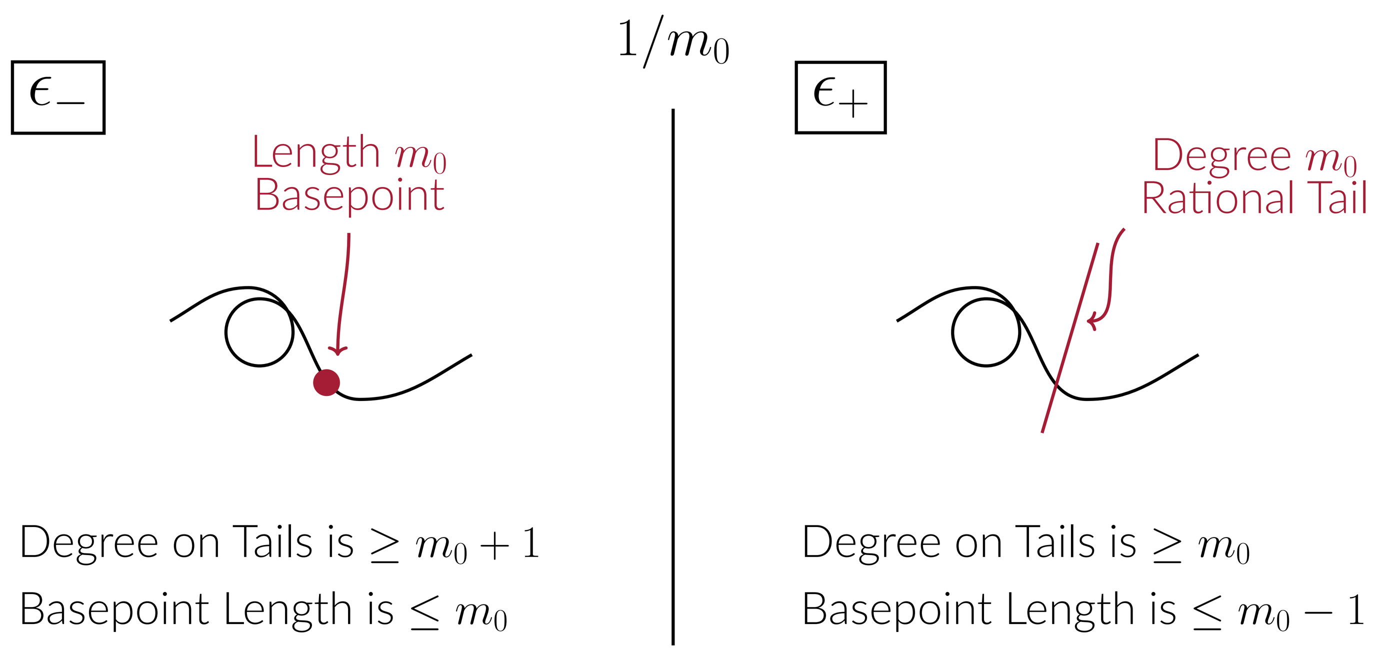
As ϵ varies from 0 to ∞, there is a wall and chamber structure on the space $\mathbb{Q}_{>0} \cup \{\infty\}$ of stability parameters ϵ. The theory changes only at discrete values of ϵ. Assuming that $(g, n) \neq (0, 0), (0, 1)$, the walls occur at $\epsilon_0 = 1/m_0$ with $m_0 \in \mathbb{N}$.



Let ϵ_- denote a rational number in the chamber $(\frac{1}{m_0+1}, \frac{1}{m_0}]$ to the left of the ϵ_0 wall. Similarly, take ϵ_+ to be a rational number in the interval $(\frac{1}{m_0}, \frac{1}{m_0-1}]$ to the right of the wall.

Geometry of Crossing an ϵ_0 Wall

As we cross a wall from the ϵ_- chamber to the ϵ_+ chamber, basepoints of length m_0 fail to be stable, and are replaced by degree m_0 rational tails.



ϵ_0-Semistable Quasimaps

Fix $\epsilon_0 = 1/m_0$. We aim to construct an algebraic stack containing both $\mathrm{QMap}^{\epsilon_-}_{g,n}(\mathbb{P}^N, d)$ and $\mathrm{QMap}^{\epsilon_+}_{g,n}(\mathbb{P}^N, d)$ as open substacks.

$$\begin{array}{ccc} & \mathrm{QMap}^{\epsilon_0}_{g,n}(\mathbb{P}^N, d) & \\ \circlearrowleft & & \circlearrowright \\ \mathrm{QMap}^{\epsilon_-}_{g,n}(\mathbb{P}^N, d) & & \mathrm{QMap}^{\epsilon_+}_{g,n}(\mathbb{P}^N, d) \end{array}$$

Definition. A quasimap is **ϵ_0-semistable** if

1. The degree of L restricted to a rational tail is $\geq m_0$.
2. The degree of L restricted to a rational bridge is strictly positive.
3. The length of a basepoint is $\leq m_0$.

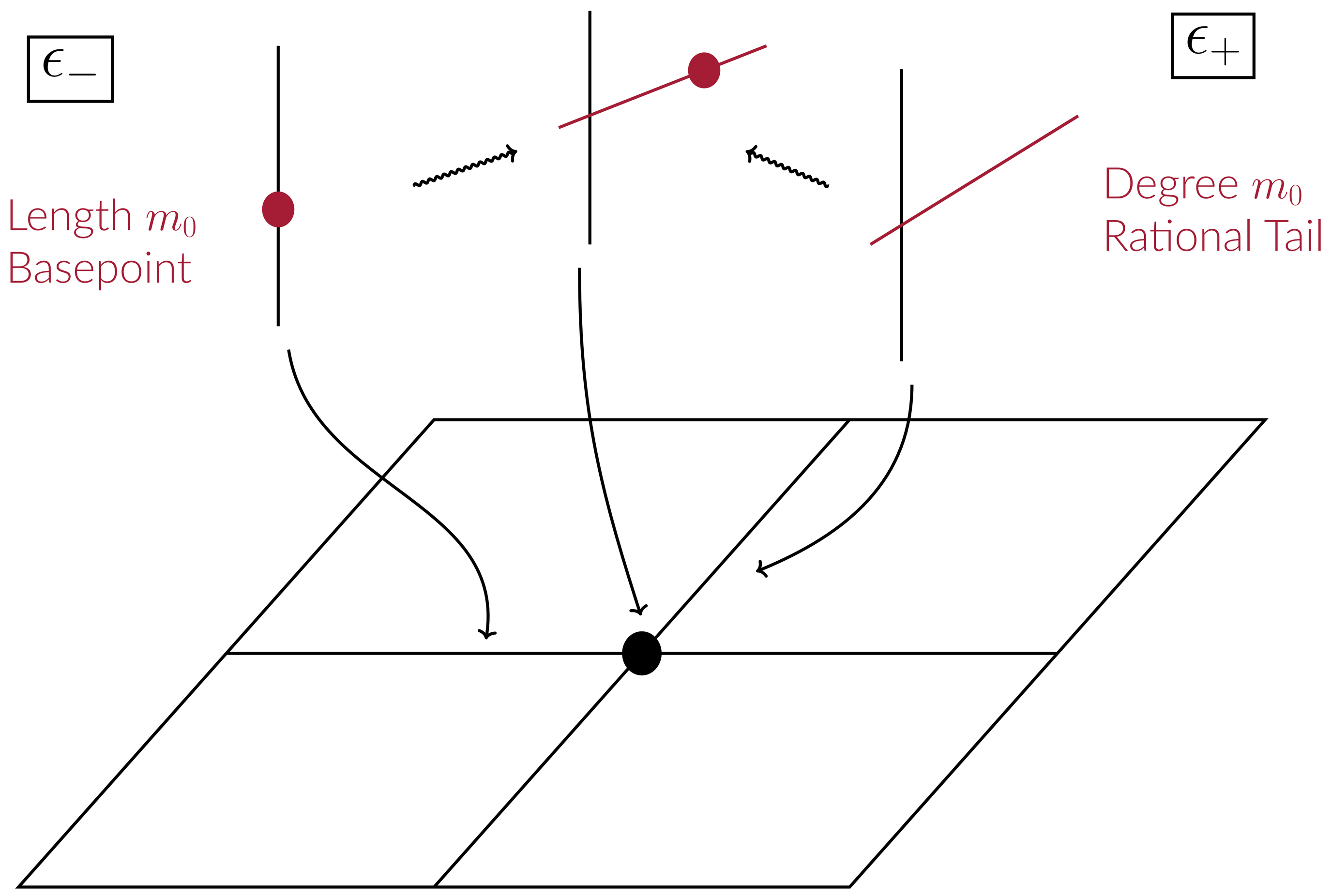
The stack $\mathrm{QMap}^{\epsilon_0}_{g,n}(\mathbb{P}^N, d)$ contains objects with infinite automorphism groups, namely quasimaps with a degree m_0 rational tail containing a length m_0 basepoint.

Theorem 1: Existence of a Good Moduli Space

Theorem [S., 2026]. The algebraic stack $\mathrm{QMap}^{\epsilon_0}_{g,n}(\mathbb{P}^N, d)$ admits a proper good moduli space.

Proof Strategy: We analyze necessary and sufficient conditions for an algebraic stack to admit a good moduli space. The algebraic stack we are interested in, $\mathrm{QMap}^{\epsilon_0}_{g,n}(\mathbb{P}^N, d)$, is defined over $\mathbb{C}$ and is of finite type with affine diagonal.

Theorem [AHLH23]. An algebraic stack of finite type over $\mathbb{C}$ with an affine diagonal admits a separated good moduli space if and only if it is Θ-reductive and S-complete.



Both Θ-reductivity and S-completeness are codimension-two filling conditions over the punctured surfaces $\Theta_R := \mathrm{Spec}(R[t])/\mathbb{G}_m$ and $\overline{\mathrm{ST}}_R := \mathrm{Spec}(R[s,t]/(st - \pi))/\mathbb{G}_m$, respectively. Here, R denotes a discrete valuation ring with maximal ideal (π).

Θ-Stratifications of $\mathrm{QMap}^{\epsilon_0}_{g,n}(\mathbb{P}^N, d)$

Definition. Let Θ denote the quotient stack $\mathbb{A}^1/\mathbb{G}_m$. A **filtration** $f : \Theta \to \mathrm{QMap}^{\epsilon_0}_{g,n}(\mathbb{P}^N, d)$ of a point of $\mathrm{QMap}^{\epsilon_0}_{g,n}(\mathbb{P}^N, d)$ is a $\mathbb{G}_m$-equivariant family of ϵ_0-semistable quasimaps over $\mathbb{A}^1$.

We want to study stability for filtrations of points of the stack $\mathrm{QMap}^{\epsilon_0}_{g,n}(\mathbb{P}^N, d)$. To do this, we use a structure on the stack called a **numerical invariant** outlined in [HL22]. Given a filtration f of a point of $\mathrm{QMap}^{\epsilon_0}_{g,n}(\mathbb{P}^N, d)$, we can study the numerical invariant

$$\nu(f) := \frac{f^*\ell}{\sqrt{f^*b}} \in \mathbb{R}$$

for $\ell \in H^2(\mathrm{QMap}^{\epsilon_0}_{g,n}(\mathbb{P}^N, d), \mathbb{Q})$ and a positive-definite norm $b \in H^4(\mathrm{QMap}^{\epsilon_0}_{g,n}(\mathbb{P}^N, d), \mathbb{Q})$. This numerical invariant comes from the data of the $\mathbb{G}_m$-action on the family of quasimaps over the central fibre.

Definition. A point of $\mathrm{QMap}^{\epsilon_0}_{g,n}(\mathbb{P}^N, d)$ is **ν-unstable** if there exists a filtration f of the point such that $\nu(f) < 0$. Otherwise, it is **ν-semistable**.

Theorem 2: Variation of Good Moduli Space

Theorem [S., 2026]. We can find cohomology classes $\ell_1, \ell_2 \in H^2(\mathrm{QMap}^{\epsilon_0}_{g,n}(\mathbb{P}^N, d), \mathbb{Q})$, and norm on graded points $b \in H^4(\mathrm{QMap}^{\epsilon_0}_{g,n}(\mathbb{P}^N, d), \mathbb{Q})$ such that the numerical invariant associated to these cohomology classes defines a Θ-stratification of $\mathrm{QMap}^{\epsilon_0}_{g,n}(\mathbb{P}^N, d)$, with the ℓ_1-semistable locus being $\mathrm{QMap}^{\epsilon_-}_{g,n}(\mathbb{P}^N, d)$ and the ℓ_2-semistable locus being $\mathrm{QMap}^{\epsilon_+}_{g,n}(\mathbb{P}^N, d)$.

Work in Progress: K-Theoretic Wall Crossing Formula

By varying the Θ-stratification of $\mathrm{QMap}^{\epsilon_0}_{g,n}(\mathbb{P}^N, d)$, we can use the non-abelian localization formula to develop a K-theoretic wall crossing formula for ϵ-stable quasimaps.

Conjecture. The wall crossing formula coming from variation of Θ-stratification on $\mathrm{QMap}^{\epsilon_0}_{g,n}(\mathbb{P}^N, d)$ recovers the K-theoretic wall-crossing formula for ϵ-stable quasimaps developed by Zhou and Zhang in [ZZ20] in the case where the target is $[\mathbb{C}^{N+1}/\mathbb{C}^*]$.

References

[AHLH23] Jarod Alper, Daniel Halpern-Leistner, and Jochen Heinloth. Existence of moduli spaces for algebraic stacks. *Inventiones mathematicae*, 234(3):949–1038, August 2023.

[CFKM14] Ionuț Ciocan-Fontanine, Bumsig Kim, and Davesh Maulik. Stable quasimaps to git quotients. *Journal of Geometry and Physics*, 75:17–47, 2014.

[HL22] Daniel Halpern-Leistner. On the structure of instability in moduli theory, 2022.

[ZZ20] Ming Zhang and Yang Zhou. K-theoretic quasimap wall-crossing, 2020.

Acknowledgments

This work was supported by a Natural Sciences and Engineering Research Council of Canada (NSERC) Doctoral Graduate Research Scholarship.

srs382@cornell.edu